4.1 - Preliminary Theory–Linear Equations

Definition: The equations

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0 \qquad (1) \quad \text{and}$$

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \qquad (2)$$

are nth-order, linear differential equations. (1) is **homogeneous**, and if $g(x) \not\equiv 0$, then (2) is **nonhomogeneous**.

Definition: The points $y(x_0) = y_0, y'(x_0) = y_1, \cdots, y^{(n-1)}(x_0) = y_{n-1}$ are **initial conditions**. An **initial-value problem** is (1) or (2), together with initial conditions. In the event that we have information such as $y(a) = y_0,\ y(b) = y_1$, then we have a **boundary-value problem**.

Theorem: Existence of a Unique Solution

Let $a_n(x), a_{n-1}(x), \ldots, a_1(x)$ and $g(x)$ be continuous on an interval I and let $a_n(x) \neq 0$ for every x in this interval. If $x = x_0$ is any point in this interval, then a solution $y(x)$ of the initial-value problem (1) exists on the interval and is unique.

Example: The given two-parameter family is a solution of the indicated differential equation on the interval $(-\infty, \infty)$. Determine whether a member of the family can be found that satisfies the boundary conditions.

$y = c_1x^2 + c_2x^4 + 3; \quad x^2y'' - 5xy' + 8y = 24$

(a) $y(-1) = 0, \quad y(1) = 4$ (b) $y(0) = 1, \quad y(1) = 2$

(c) $y(0) = 3, \quad y(1) = 0$ (d) $y(1) = 3, \quad y(2) = 15$

Definition: A set of functions $f_1(x), f_2(x), \ldots, f_n(x)$ is said to be **linearly dependent** on an interval I if there exist constants $c_1, c_2, \ldots, c_n$, not all zero, such that $c_1 f_1(x) + c_2 f_2(x) + \cdots + c_n f_n(x) = 0$ for every x in the interval. If the set of functions is not linearly dependent on the interval, it is said to be **linearly independent**

Definition: Suppose each of the functions $f_1(x), f_2(x), \ldots, f_n(x)$ possesses at least $n-1$ derivatives. The determinant

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix},$$

where the primes denote derivatives, is called the **Wronskian** of the functions

Theorem: **Criterion for Linearly Independent Solutions**

Let $y_1, y_2, \dots, y_n$ be n solutions of the homogeneous nth-order differential equation (1) on an interval I. Then the set of solutions is **linearly independent** on I if and only if $W(y_1, y_2, \dots, y_n) \neq 0$ for every x in the interval.

Example: Determine whether the given set of functions is linearly independent on the interval $(-\infty, \infty)$.

$f_1(x) = \cos 2x,\ f_2(x) = 1,\ f_3(x) = \cos^2 x$

Theorem: **Superposition Principle–Homogeneous Equations**

Let $y_1, y_2, \dots, y_k$, be solutions of the homogeneous nth-order differential equation (1) on an interval I. Then the linear combination $c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)$, where the c_i, $i = 1, 2, \dots, k$ are arbitrary constants, is also a solution on the interval.

Definition: Any set $y_1, y_2, \ldots, y_n$, of n of linearly independent solutions of the homogeneous nth-order differential equation (1) on an interval I is said to be a **fundamental set of solutions** on the interval.

Theorem: General Solution–Homogeneous Equations

Let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of the homogeneous linear nth-order differential equation (1) on an interval I. Then the **general solution** of the equation on the interval is $y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$.

Example: Verify that the given functions form a fundamental set of solutions of the differential equation on the indicated interval. Form the general solution.

$x^2 y'' + xy' + y = 0; \quad \cos(\ln x), \sin(\ln x), (0, \infty)$

Theorem: **Existence of a Fundamental Set**

There exists a fundamental set of solutions for the homogenous linear nth-order differential equation (1) on an interval I.

Definition: Any function y_p, free of arbitrary parameters, that satisfies (2) is said to be a **particular solution** of the equation.

Theorem: **General Solution–Nonhomogeneous Equations**

Let y_p be any particular solution of the nonhomogeneous linear nth-order differential equation (2) on an interval I and let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of the associated homogeneous differential equation (1) on I. Then the **general solution** of the equation on the interval is
$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p(x)$, where the c_i, $i = 1, 2, \ldots, n$ are arbitrary constants.

Theorem: **Superposition Principle–Nonhomogeneous Equations**

Let $y_{p_1}, y_{p_2}, \ldots, y_{p_k}$ be k particular solutions of the nonhomogeneous linear nth-order differential equation (2) on an interval I corresponding, in turn, to k distinct functions $g_1, g_2, \ldots, g_k$. Then $y_p(x) = y_{p_1}(x) + y_{p_2}(x) + \ldots + y_{p_k}(x)$ is a particular solution of
$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g_1(x) + g_2(x) + \ldots + g_k(x)$

Example: (a) Verify that $y_{p_1} = 3e^{2x}$ and $y_{p_2} = x^2 + 3x$, respectively are particular solutions of

$y'' - 6y' + 5y = -9e^{2x}$ and

$y'' - 6y' + 5y = 5x^2 + 3x - 16.$

(b) Use part (a) to find particular solutions of

$y'' - 6y' + 5y = 5x^2 + 3x - 16 - 9e^{2x}.$